CSUC
Department of Physics
301A Mechanics

I. *MATH TOOLS: WEEK 1. PATTERNS OF GROWTH*

One key math-tool will be our knowledge of how functions grow and dominate each other. All experienced practitioners carry about in their heads a "hierarchy of growth". For example, you are familiar with the idea that higher powers dominate lower powers, by which we mean:

$$\lim_{x \to \infty} \frac{x^m}{x^n} \; = \; 0$$

for any non-negative $\{m, n\}$ as long as we have $n \; > \; m$.

But you may not be quite so aware that there are many many more of these relations "out there" that are very useful. One lovely observation is to be seen in the following graph. We observe that for any non-negative n and for any $x \; > \; 1$, we always have $x^n \; > \; 1$. Actually we can say a bit more. For any positive number n, no matter how

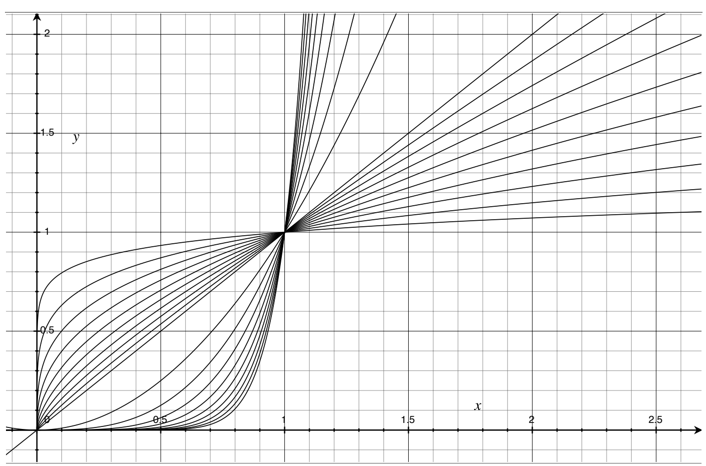

FIG. 1: Powers from .1 to 10

small or large, and for positive x values, we have both:

$$\lim_{x \to \infty} x^n \; = \; \infty$$

$$\lim_{x \to 0} x^n \; = \; 0$$

Why is this useful ? Well, consider this, for example. Since $x > 1$ implies we always have $x^n > 1$, if we then divide by x, we can conclude $x^{n-1} > 1 / x$. Further, if we integrate this relationship from 1 to x, we can conclude:

$$\int_1^x dx\, x^{n-1} > \int_1^x dx\, 1 / x = \log(x)$$

So, we can now conclude:

$$\frac{1}{n} (x^n - 1) > \log(x)$$

and so, surely we have:

$$\frac{1}{n} x^n > \log(x)$$

Now divide the whole equation by $x^{n+\epsilon}$ where ϵ is <u>any</u> non-negative number no matter how small, and we achieve:

$$\frac{1}{n\, x^{\epsilon}} > \frac{\log(x)}{x^{n+\epsilon}}$$

Since the left hand side tends to zero as x becomes large, and since both n and ϵ were arbitrary positive numbers and thus their sum $p \equiv (n + \epsilon)$ is also arbitrary, we conclude that for <u>any</u> positive number p, no matter how small, we must have :

$$\lim_{x\to\infty} \frac{\log (x)}{x^p} = 0$$

So apparently *any* positive power (no matter how small) beats a logarithm ! The log function is the universal loser ! It's as close to being constant as you can be, without being constant. In fact we can take this one step further. Since any positive power of a positive number tending to zero must also tend to zero (see above), it must be the case that if q is any positive number, that:

$$\lim_{x\to\infty} \left(\frac{\log (x)}{x^p} \right)^q = 0$$

Since both p and q are arbitrary positive numbers, then so is their product which we may call r. Thus, finally, we may conclude that for any positive q and any positive r we have:

$$\lim_{x\to\infty} \frac{(\log (x))^q}{x^r} = 0$$

This expression admits even one further transformation if we write $x = e^t$, for then our equation reads:

$$\lim_{t\to\infty} \frac{t^q}{e^{t\,r}} = 0$$

We summarize by saying:

" any positive exponential beats any positive algebraic power which beats any positive power of a logarithm" .